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# A 'win–win' mechanism for low-drag transients in controlled two-dimensional channel flow and its implications for sustained drag reduction

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A simple pressure-based feedback control strategy for wall-transpiration control of incompressible unsteady two-dimensional channel flow was recently investigated by Aamo, Krstic & Bewley (2003). Nonlinear two-dimensional channel flow simulations which implemented this control strategy resulted in flow transients with instantaneous drag far lower than that of the corresponding laminar flow. The present article examines the physical mechanism by which this very low level of instantaneous drag was attained. It then explores the possibility of achieving sustained drag reductions to below the laminar level by initiating such low-drag transients on a periodic basis. All attempts at sustaining the mean flow drag below the laminar level fail, perhaps providing indirect evidence in favour of the conjecture that the laminar state might provide a fundamental 'performance limitation' in such flows. Mathematical analysis of two-dimensional and three-dimensional channel-flow systems establishes a direct link between the average drag increase due to flow-field unsteadiness and a weighted space/time average of the Reynolds stress. Phenomenological justification of the conjecture is provided by a Reynolds analogy between convective momentum transport and convective heat transport. Proof of the conjecture remains an open problem.

## 1. An open question of fundamental significance

Motivated by a desire to quantify possible fundamental performance limitations and to understand better certain proposed mechanisms for channel-flow drag reduction, the following, as yet unproven, conjecture was formally proposed in Bewley (2001):

Conjecture: *The lowest sustainable drag of an incompressible constant mass-flux channel flow, when controlled via a distribution of zero-net mass-flux blowing/suction over the no-slip channel walls, is exactly that of the laminar flow.*

Note that, by the 'sustainable drag' (denoted $\langle D\rangle_\infty$), we mean the time average (denoted $\langle D\rangle_T$) of the instantaneous drag (denoted $D(t)$) as the averaging time $T$ approaches infinity, i.e.

$$\langle D\rangle_\infty \triangleq \lim_{T\to\infty}\langle D\rangle_T \triangleq \lim_{T\to\infty}\frac{1}{T}\int_0^T D(t)\,\mathrm{d}t \triangleq \lim_{T\to\infty}\frac{-\mu}{T}\int_0^T\int_{\Gamma_2^\pm}\frac{\partial u}{\partial n}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

where $\boldsymbol{n}$ is an outward facing normal, $\Gamma_2^\pm$ denotes the set given by the union of the upper and lower walls of the channel, $\mu$ is the viscocity, $u$ is the streamwise

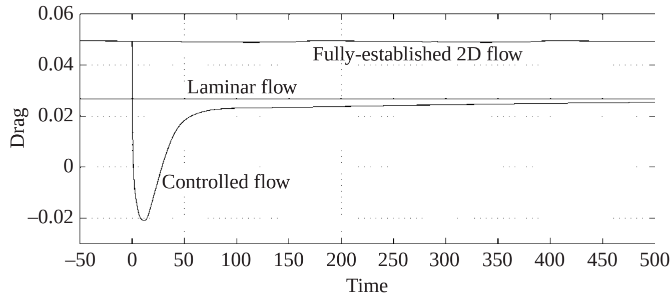


FIGURE 1. History of drag. In the controlled flow, the simulation is initiated from fully established unsteady two-dimensional flow at $Re = 7500$, and the stabilizing pressure-based feedback control strategy (2.1) with $k = 0.125$ is turned on at $t = 0$. The drag of the laminar flow and the fully established unsteady two-dimensional channel flow (Jimenez 1990) are shown for comparison.

component of the velocity vector $\boldsymbol{u}$, and $D_L$ denotes the drag of the corresponding laminar channel flow with the same dimensions, viscosity and bulk velocity.

Recent two-dimensional simulations of controlled channel flows demonstrating strong $D(t) < D_L$ transients (Cortelezzi *et al.* 1998; Aamo, Krstić & Bewley 2003) have cast some doubt as to the validity of this conjecture. The purpose of this note is to investigate the mechanism behind these transients and the possible use of this mechanism to provide sustained drag reductions to sublaminar levels.

## 2. The transient low-drag mechanism

The following feedback control rule was proposed and tested in Aamo *et al.* (2003):

$$\phi^{\pm} = k\,(p^{\pm} - p^{\mp}), \tag{2.1}$$

where $\phi^{\pm} = -\boldsymbol{u}\cdot\boldsymbol{n}$ is the blowing/suction distribution which is applied to the walls $\Gamma_2^{\pm}$ of the channel flow system as the control, $p^{\pm}$ is the pressure on the corresponding wall, $p^{\mp}$ denotes the pressure on the opposite wall, and $k$ is a constant. With such a strategy, blowing at one wall of the channel is always countered with suction of equal magnitude at the opposite wall. A feedback rule of this form was motivated by Lyapunov analysis of the kinetic energy of the channel flow system integrated over the entire channel, integrated by parts, and examined at extremely low Reynolds number (see Aamo *et al.* 2003 for details).

Regardless of the motivation for considering the feedback rule (2.1), it is of interest here to study the flow that results when (2.1) is applied to the two-dimensional channel flow system at supercritical Reynolds numbers. It was observed in Aamo *et al.* (2003) that feedback of this type, when applied to the fully established unsteady flow in a two-dimensional channel at $Re = 7500$, resulted in a flow transient with drag far below the laminar level. A similar transient was also observed in earlier work by Cortelezzi *et al.* (1998), where a low-drag transient to 50% below the laminar level was reported in a two-dimensional flow.

As reported in figures 1–4, a transient which actually achieves negative total drag for a short period of time is achieved by applying (2.1) to a fully-established, unsteady, constant mass-flux two-dimensional channel flow at $Re = 7500$. Jiménez (1990) describes the uncontrolled two-dimensional flow system. The simulations

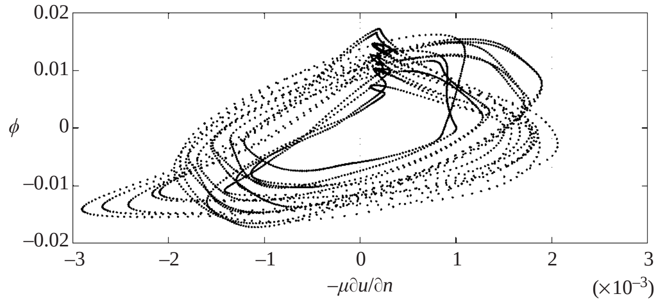


FIGURE 2. Scatterplot of $\phi$ versus $(-\mu\partial u/\partial n)$ at $t=5$.

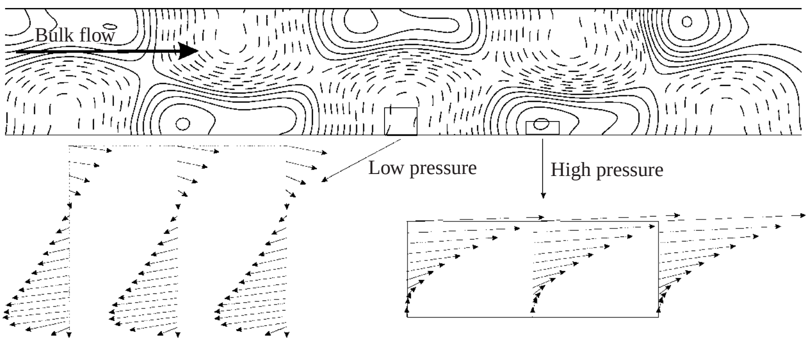


FIGURE 3. Win–win mechanism at $t=5$: intensification of local regions of negative drag by suction in low-pressure regions and moderation of positive drag by blowing in high-pressure regions. Shown are contours of pressure in 1/6 of the computational domain (top) and selected velocity profiles (bottom).

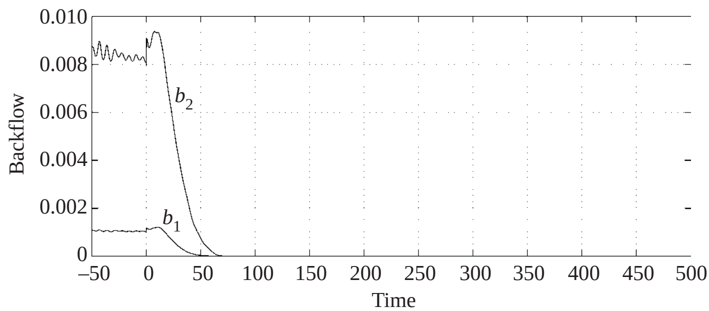


FIGURE 4. Elimination of backflow after control is turned on, as measured by $b_1(t)$ and $b_2(t)$.

reported here used a box length of 60 times the channel half-width at a resolution of $1024 \times 128$ using the DNS code of Lumley & Blossey (1998).

The flow at $t=0^-$ in figure 1, a fully established unsteady flow in a two-dimensional channel, has extensive regions of backflow near the walls. This appears to be the key to initiating a $D(t) < D_L$ transient. A scatter plot of the local control $\phi$ as a function of the local value of drag $(-\mu\partial u/\partial n)$ at $t=5$ (shortly after the control is turned on) is shown in figure 2, demonstrating correlation of blowing with local regions of positive drag and suction with local regions of negative drag using the present strategy (76% of the samples are in the first and third quadrants). By generally applying suction at the walls in regions of negative drag, and applying blowing in regions of large positive drag, the negative drag regions are intensified (locally, more negative drag) and the high positive drag regions are moderated (locally, less positive drag), as illustrated in figure 3. In terms of reducing the total instantaneous drag $D(t)$ integrated over the walls at time $t = 5$, both effects are beneficial, and thus the control application results, for a brief period of time, in a 'win–win' situation, facilitating a transient reduction in skin-friction drag to well below laminar levels. Unfortunately, the wall suction quickly acts to remove the backflow from the flow domain entirely, after which the instantaneous drag $D(t)$ asymptotes back to the laminar level $D_L$.

A metric which quantifies the backflow present at any instant in a particular flow is given by $b_p = [(1/V)\int_{\Omega^-} |u|^p \, \mathrm{d}\boldsymbol{x}]^{1/p}$, where $\Omega^-$ is the subset of the channel flow domain $\Omega$ which is characterized by regions of flow with negative streamwise velocity, i.e. $\Omega^- = \{\Omega(x, y) | u(x, y) < 0\}$, and $V$ is the volume of the entire channel domain $\Omega$. For the simulation depicted in figures 1–3, plots of the history of $b_1$ and $b_2$ are shown in figure 4. Note that, by both measures, the backflow is quickly eliminated after the control is initiated; flow visualizations such as figure 3 demonstrate that the backflowing fluid in $\Omega^-$ is simply removed from the channel by the control suction.

## 3. Cycling the controller off and on

As a 'standard' problem to test the capability of a given control strategy for reducing time-averaged drag to below laminar levels, a series of controlled two-dimensional channel-flow simulations at $Re = 7500$ were initialized from small (random) perturbations to a laminar flow profile. The control producing the $D(t) < D_L$ transients was cycled off and on periodically, with the 'running time average' of the drag, $\langle D \rangle_t = (1/t)\int_0^t D(t')\,\mathrm{d}t'$, computed as the flow evolved to quantify progress towards sustained drag increase or reduction as compared with the reference drag of the laminar flow. By initializing the test as a small perturbation of the reference (laminar) flow, the tendency of the control strategy to increase or decrease the drag as compared with the reference value is readily determined. A large variety of different periods, duty cycles and control amplitudes were explored; table 1 summarizes some of the specific cases examined in detail.

Cases 1–5 reported in table 1 were executed at a cycle time of $T_{cycle} = 3000$ for a variety of duty cycles with relatively strong stabilizing feedback applied during the second segment of each cycle. Cases 6–8 were similar, but applied relatively weak stabilizing feedback. Cases 9–13 returned to the relatively strong stabilizing feedback, but investigated a shorter cycle time. Finally, cases 14–16 were executed with destabilizing feedback applied during the first segment of each cycle, and stabilizing feedback applied during the second segment of each cycle; this was done to accelerate the formation of the backflow regions. Histories of the $L^2$ energy, the instantaneous and 'running time-averaged' drag $D(t)$ and $\langle D \rangle_t$, and the backflow measures $b_1$ and $b_2$ are illustrated in figure 5 for four representative cases.

It was found in cases 1, 2, 9, 10 and 14, with $T_2/T_1$ relatively large, that the stabilization provided by the control during the second segment of each cycle was sufficient to stabilize the entire channel flow back to the parabolic profile; to illustrate, cases 14 and 15 are plotted in figures 5($c$) and 5($d$). These cases imply that $T_2/T_1$ must be sufficiently small in order to allow a quasi-periodic limit cycle to be sustained.

| Case | $T_{cycle}$ | $T_1$ | $T_2$ | $k_1$ | $k_2$ | $\langle D\rangle_\infty/D_L$ |
|---|---|---|---|---|---|---|
| 1 | 3000 | 2600 | 400 | 0 | 0.125 | 1.00 |
| 2 | 3000 | 2700 | 300 | 0 | 0.125 | 1.00 |
| 3 | 3000 | 2800 | 200 | 0 | 0.125 | 1.09 |
| 4 | 3000 | 2900 | 100 | 0 | 0.125 | 1.20 |
| 5 | 3000 | 2950 | 50 | 0 | 0.125 | 1.42 |
| 6 | 3000 | 2000 | 1000 | 0 | 0.031 | 1.17 |
| 7 | 3000 | 2500 | 500 | 0 | 0.031 | 1.35 |
| 8 | 3000 | 2800 | 200 | 0 | 0.031 | 1.43 |
| 9 | 2000 | 1600 | 400 | 0 | 0.125 | 1.00 |
| 10 | 2000 | 1700 | 300 | 0 | 0.125 | 1.00 |
| 11 | 2000 | 1800 | 200 | 0 | 0.125 | 1.05 |
| 12 | 2000 | 1900 | 100 | 0 | 0.125 | 1.17 |
| 13 | 2000 | 1950 | 50 | 0 | 0.125 | 1.38 |
| 14 | 1000 | 325 | 675 | −0.031 | 0.031 | 1.00 |
| 15 | 1000 | 350 | 650 | −0.031 | 0.031 | 1.56 |
| 16 | 1000 | 500 | 500 | −0.031 | 0.031 | 1.84 |

TABLE 1. Representative forcing schedules explored during the parametric study: $T_{cycle}$ indicates the period of the cycle used (in units of $\delta/U_c$), $T_1$ denotes the duration of the first segment of the cycle, $T_2$ denotes the duration of the second segment, $k_1$ denotes the feedback coefficient used during the first segment, and $k_2$ denotes the feedback coefficient during the second segment. The last column indicates the time asymptotic value of the running time-average drag; note that some cases relaminarize, leading to the drag of the laminar flow, wheras other cases lead to a limit-cycling behaviour with an average drag greater than that of the laminar flow. All simulations were initialized from a slightly perturbed laminar flow. Note that $\delta$ is the channel half-width and $U_c$ is the centreline velocity of the corresponding laminar flow.

It was found in cases 5, 8, 13 and 16, with $T_2/T_1$ relatively small, that the uncontrolled (or, in case 16, destabilized) evolution of the flow during the first segment of each cycle was sufficient to drive the time-averaged drag to heightened levels.

A tradeoff is thus identified: we need $T_2/T_1$ to be sufficiently small so that there is an adequate amount of backflow to exploit during each cycle (so the ensuing transient will have a significant $D(t) < D_L$ minimum), but we also need $T_2/T_1$ to be sufficiently large so that the mean drag is not pulled up too high above the laminar level during the segment of each cycle in which the backflow is developing. Intermediate values of $T_2/T_1$ were thus sought for a variety of cycle times $T_{cycle} = T_1 + T_2$ and forcing amplitudes $k_1$ and $k_2$ over a parametric study of several simulations, some of which are reported here. Over all these simulations, this tradeoff was evident, and not once did the running average, $\langle D\rangle_t$, dip below the laminar value, $D_L$, when the simulations were initiated from the perturbed laminar state. These results indicate that it appears always to be necessary to pay a more expensive price (in terms of the time-averaged drag) to obtain the backflow than the benefit (in terms of the time-averaged drag) that can be obtained by applying suction to the backflow regions.

## 4. Analysis

For generality of this discussion, we will now analyse mathematically the three-dimensional case depicted in figure 6. The two-dimensional case may be considered as a special case of the analysis presented here.

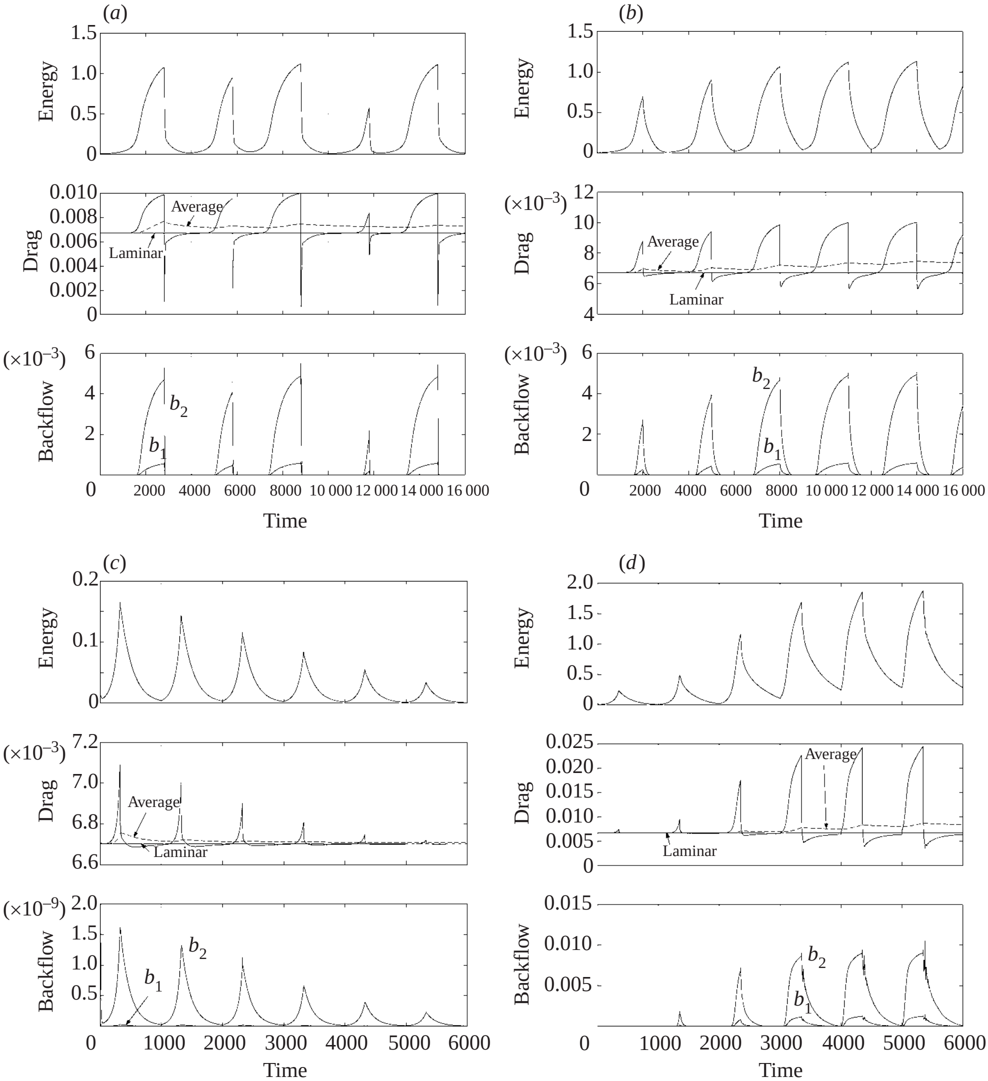


FIGURE 5. Representative time histories of parametric study, results of which are summarized in table 1. (*a*) Case 3: $T_{cycle} = 3000$, no feedback for $T_1 = 2600$, relatively strong stabilizing feedback for $T_2 = 400$. (*b*) Case 6: $T_{cycle} = 3000$, no feedback for $T_1 = 2000$, relatively weak stabilizing feedback for $T_2 = 1000$. (*c*) Case 14: $T_{cycle} = 1000$, destabilizing feedback for $T_1 = 325$, stabilizing feedback for $T_2 = 675$. (*d*) Case 15: $T_{cycle} = 1000$, destabilizing feedback for $T_1 = 350$, stabilizing feedback for $T_2 = 650$.

### 4.1. *Problem statement*

Consider the incompressible Navier–Stokes equation†

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \nu\Delta\boldsymbol{u} + \boldsymbol{i}P_x, \qquad \nabla\cdot\boldsymbol{u} = 0, \tag{4.1}$$

† Without loss of generality, we take the density as unity, so $\nu = \mu$ in the remainder of this discussion.

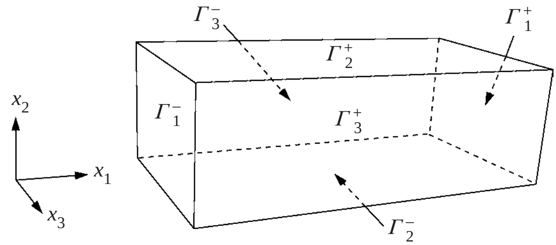


FIGURE 6. Flow domain $\Omega$ under consideration in §4, and the notation used.

governing the flow in the rectangular domain $\Omega$ of size $(0, L_x) \times (-1, 1) \times (0, L_z)$, shown in figure 6. The mean pressure gradient $P_x(t)$ in the streamwise direction $\boldsymbol{i}$ is adjusted in such a way as to maintain a constant bulk velocity

$$U_B = \frac{1}{V}\int_\Omega u_1(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x} = \text{constant} \qquad \forall t, \tag{4.2}$$

where $V = 2L_xL_z$. The initial conditions on the velocity field $\boldsymbol{u}(\boldsymbol{x}, 0)$ in the domain $\Omega$ are taken initially as the laminar flow profile

$$\boldsymbol{u}(\boldsymbol{x}, 0) = C\left(1 - x_2^2\right)\boldsymbol{i}, \tag{4.3}$$

where $C = (3/2)U_B$. The boundary conditions on the walls $(\Gamma_2^\pm)$ are no slip in the $x_1$ and $x_3$ directions and some (as yet undetermined) unsteady distribution of blowing/suction in the wall-normal $(x_2)$ direction

$$\boldsymbol{u} = -\boldsymbol{n}\,\phi(\boldsymbol{x}, t) \quad \text{on} \quad \Gamma_2^\pm, \tag{4.4}$$

with periodic boundary conditions assumed in the $x_1$- and $x_3$-directions. As $\boldsymbol{n}$ is defined as an outward facing normal, positive $\phi$ corresponds to blowing and negative $\phi$ corresponds to suction. The control is constrained to apply zero-net mass flux at each instant:

$$\int_{\Gamma_2^-} \phi\,\mathrm{d}\boldsymbol{x} = \int_{\Gamma_2^+} \phi\,\mathrm{d}\boldsymbol{x} = 0 \qquad \forall t. \tag{4.5}$$

The conjecture considered in the present paper is the assertion that, for any distribution of controls $\phi$ satisfying (4.5), the minimum sustainable drag is exactly that of the laminar flow.

### 4.2. *Relationship of conjecture to the energy dissipation and the Reynolds stress*

We are interested in time-averaged quantities defined such that

$$\langle f\rangle_T = \frac{1}{T}\int_0^T f\,\mathrm{d}t;$$

in particular, we are interested in the $T \to \infty$ limit of such quantities, which we denote $\langle f\rangle_\infty$. The $L_2$ norm of a function $g(\boldsymbol{x})$ is denoted

$$\|g\|_2 = \left(\int_\Omega |g(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\right)^{1/2}.$$

Of particular interest is the instantaneous energy dissipation rate, given by

$$\nu \, \|\nabla \boldsymbol{u}\|_2^2 = \nu \sum_{\iota,\kappa=1}^{3} \left\| \frac{\partial u_\iota}{\partial x_\kappa} \right\|_2^2 .$$

Note that it is easily shown (by integrating the $x_1$-component of the Navier–Stokes equation) that

$$\langle D \rangle_T \triangleq \langle P_x \rangle_T = \frac{1}{T} \left[ \int_0^T \int_{\Gamma_2^-} \nu \frac{\partial u_1}{\partial x_2} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t - \int_0^T \int_{\Gamma_2^+} \nu \frac{\partial u_1}{\partial x_2} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \right]. \tag{4.6}$$

That is, $\langle D \rangle_T \triangleq \langle P_x \rangle_T$ is simply the skin-friction drag of the flow integrated over both walls $\Gamma_2^\pm$ and averaged over the time interval $[0, T]$.

A relationship between the average skin-friction drag $\langle D \rangle_\infty$ and the average energy dissipation rate $\langle \nu \, \|\nabla \boldsymbol{u}\|_2^2 \rangle_\infty$ is now determined by taking the scalar product of the Navier–Stokes equation (4.1) with the velocity field $\boldsymbol{u}$, integrating over space, integrating by parts, and noting that $\boldsymbol{u} \cdot \partial \boldsymbol{u} / \partial n = 0$, which gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \, \|\boldsymbol{u}\|_2^2 + \int_\Omega u_i u_j \frac{\partial u_i}{\partial x_j} \, \mathrm{d}\boldsymbol{x} + \int_\Omega u_i \frac{\partial p}{\partial x_i} \, \mathrm{d}\boldsymbol{x} + \nu \, \|\nabla \boldsymbol{u}\|_2^2 = P_x U_B V,$$

with summation notation implied. Integrating the second and third terms on the left-hand side by parts, applying continuity and the boundary conditions, gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \, \|\boldsymbol{u}\|_2^2 + \nu \, \|\nabla \boldsymbol{u}\|_2^2 - \int_{\Gamma_2^\pm} \phi \left( \frac{\phi^2}{2} + p \right) \mathrm{d}\boldsymbol{x} = P_x U_B V.$$

Taking the time average in the limit that $T \to \infty$, assuming *a priori* that $\|\boldsymbol{u}\|_2^2$ remains bounded (see, for example, Constantin & Doering 1994), gives

$$\boxed{\langle D \rangle_\infty = \frac{1}{U_B V} \left[ \langle \nu \, \|\nabla \boldsymbol{u}\|_2^2 \rangle_\infty - \left\langle \int_{\Gamma_2^\pm} \phi \left( \frac{\phi^2}{2} + p \right) \mathrm{d}\boldsymbol{x} \right\rangle_\infty \right].} \tag{4.7}$$

A relationship between the average skin-friction drag $\langle D \rangle_\infty$ and the Reynolds stress is now determined. Decompose $\boldsymbol{u}$ according to $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{i} U(x_2) + \boldsymbol{v}(\boldsymbol{x}, t)$, where we will take $U(x_2)$ as the parabolic laminar flow profile $U(x_2) = C(1 - x_2^2)$ and therefore $\boldsymbol{v}(\boldsymbol{x}, 0) = 0$ and $\boldsymbol{v} = -\boldsymbol{n} \, \phi(\boldsymbol{x}, t)$ on $\Gamma_2^\pm$. Note that it follows immediately from (4.2) that

$$\int_\Omega v_1(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x} = 0 \qquad \forall t \tag{4.8}$$

and, by substitution of $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{i} U(x_2)$ and $\phi = 0$ into (4.7), that

$$D_L U_B V = \nu L_x L_z \int_{-1}^{1} U'^2 \, \mathrm{d}\boldsymbol{x}, \tag{4.9}$$

where $D_L$ is the drag of the laminar flow $U$. Substituting $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{i} U(x_2) + \boldsymbol{v}(\boldsymbol{x}, t)$ into (4.1), noting that the laminar flow $U(x_2)$ itself satisfies (4.1) for some mean pressure gradient $P_{x,L}$, it follows that

$$\left. \begin{aligned} &\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + U \frac{\partial \boldsymbol{v}}{\partial x_1} + \boldsymbol{i} v_2 U' + \nabla p = \nu \Delta \boldsymbol{v} + \boldsymbol{i}(P_x - P_{x,L}), \\ &\nabla \cdot \boldsymbol{v} = 0. \end{aligned} \right\} \tag{4.10}$$

Going through a similar development as that which led to (4.7) and applying (4.8) leads to

$$0 = \langle \nu \, \|\nabla \boldsymbol{v}\|_2^2 \rangle_\infty - \left\langle \int_{\Gamma_2^\pm} \phi \left( \frac{\phi^2}{2} + p \right) \mathrm{d}\boldsymbol{x} \right\rangle_\infty + \left\langle \nu \int_\Omega U' \frac{\partial v_1}{\partial x_2} \, \mathrm{d}\boldsymbol{x} \right\rangle_\infty + \left\langle \int_\Omega v_1 v_2 U' \, \mathrm{d}\boldsymbol{x} \right\rangle_\infty . \quad (4.11)$$

Note the fact that, since $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{i} U(x_2) + \boldsymbol{v}(\boldsymbol{x}, t)$, it follows that

$$\|\nabla \boldsymbol{u}\|_2^2 = L_x L_z \int_{-1}^{1} U'^2 \, \mathrm{d}x_2 + 2 \int_\Omega U' \frac{\partial v_1}{\partial x_2} \, \mathrm{d}\boldsymbol{x} + \|\nabla \boldsymbol{v}\|_2^2 . \quad (4.12)$$

Further, integration by parts, applying the boundry conditions on $\boldsymbol{v}$, noting that $U$ is quadratic, and applying (4.8) shows that

$$\int_\Omega U' \frac{\partial v_1}{\partial x_2} \, \mathrm{d}\boldsymbol{x} = -\int_\Omega U'' v_1 \, \mathrm{d}\boldsymbol{x} = -U'' \int_\Omega v_1 \, \mathrm{d}\boldsymbol{x} = 0. \quad (4.13)$$

Finally, combining (4.7), (4.9), (4.11), (4.12) and (4.13) gives simply

$$\boxed{\langle D \rangle_\infty = D_L - \frac{1}{U_B V} \left\langle \int_\Omega v_1 v_2 U' \, \mathrm{d}\boldsymbol{x} \right\rangle_\infty \triangleq D_L + \frac{1}{U_B V} \langle H \rangle_\infty .} \quad (4.14)$$

Note that the $\phi(\phi^2/2 + p)$ term representing the direct influence of the control $\phi$ on the energy balance in the flow cancels from this expression. The only way the control can affect the drag is indirectly, via the effect of the term involving the average value of $v_1 v_2 U'$.

The question of whether or not drag can be maintained below laminar levels thus boils down to a question of whether or not the time average of the quantity $H = -\int_\Omega v_1 \, v_2 \, U' \, \mathrm{d}\boldsymbol{x}$ can be made negative by the action of a control boundary condition $\boldsymbol{v} = -\boldsymbol{n} \phi$ on $\Gamma_2^\pm$, where $\phi$ satisfies (4.5). Note the relationship of the quantity $H$ to the Reynolds stress of a channel flow. In an uncontrolled flow ($\phi = 0$) when self-sustained unsteadiness (turbulence) is present (i.e. when $\boldsymbol{v} \neq \boldsymbol{0}$), the time average of this term is positive, resulting in the substantial drag increase seen in turbulent flows. The question considered in this paper is whether or not, by action of control, the near-wall unsteadiness may be 'fundamentally restructured' in such a way that $\langle H \rangle_\infty$ is negative.

### 4.3. *An equivalent eigenvalue problem*

In the steady case, the question of whether or not the term $H$ is positive in the nonlinear Navier–Stokes problem (even when control is applied) is related to the self-adjoint generalized eigenvalue problem

$$\lambda \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \pi \end{pmatrix} = \begin{pmatrix} 0 & -U'(x_2) & 0 & \partial/\partial x_1 \\ -U'(x_2) & 0 & 0 & \partial/\partial x_2 \\ 0 & 0 & 0 & \partial/\partial x_3 \\ -\partial/\partial x_1 & -\partial/\partial x_2 & -\partial/\partial x_3 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \pi \end{pmatrix} \quad (4.15)$$

with boundary conditions

$$\boldsymbol{w} = -\boldsymbol{n} \, \phi(\boldsymbol{x}, t) \quad \text{on} \quad \Gamma_2^\pm, \quad (4.16)$$

where $\phi$ satisfies (4.5). Note that $\boldsymbol{w}$ and $\pi$ are periodic in the streamwise and spanwise directions. By premultiplying (4.15) with $\begin{pmatrix} w_1 & w_2 & w_3 & \pi \end{pmatrix}$, integrating over the domain $\Omega$, integrating by parts, and applying continuty, we see that

$$-\int_{\Omega} w_1\, w_2\, U'\, \mathrm{d}\boldsymbol{x} = \tfrac{1}{2}\lambda\, \|\boldsymbol{w}\|_2^2 + \int_{\Gamma_2^{\pm}} \pi\, \phi\, \mathrm{d}\boldsymbol{x}.$$

By the first line of (4.15) evaluated at the wall, the boundary condition (4.16), the fact that $U'(x_2) = -2Cx_2$, and periodicity of $\pi$, it follows that

$$\int_{\Gamma_2^{\pm}} \pi\, \phi\, \mathrm{d}\boldsymbol{x} = \int_{\Gamma_2^{\pm}} \pi \left(\frac{1}{2C}\frac{\partial \pi}{\partial x_1}\right) \mathrm{d}\boldsymbol{x} = \frac{1}{4C}\int_{\Gamma_2^{\pm}} \frac{\partial \pi^2}{\partial x_1}\, \mathrm{d}\boldsymbol{x} = 0,$$

and thus, taking $\boldsymbol{v} = \boldsymbol{w}$ in (4.14),

$$\langle D(\boldsymbol{w})\rangle_{\infty} = D_L + \frac{\lambda}{2U_B V}\, \|\boldsymbol{w}\|_2^2\,.$$

Since drag is linear in the velocity field, the present conjecture (in the steady case) is equivalent to the statement that the set of eigenvectors $\boldsymbol{w}$ corresponding to the positive eigenvalues $\lambda$ of (4.15) form a complete basis of all possible steady-flow solutions of (4.1)–(4.4) for arbitrary $\phi$ satisfying (4.5). Note that equation (4.15) is a construed, but perhaps useful, test equation and is not the equation obtained via linearization of the governing equations; as motivated by the work of Constantin & Doering (1994), further analysis of this equivalent eigenvalue problem provides a promising avenue for mathematical proof of the present conjecture.

### 4.4. *Phenomenological justification*

Yet another way of interpreting the present conjecture is that, on average (that is, averaged in space in steady problems and averaged in both space and time in unsteady problems), convection triggered by unsteadiness or non-uniformity of the boundary conditions can only accelerate momentum transfer in the direction of viscous diffusion, and cannot be used to counter this effect.

Phenomenological justification of the conjecture is provided by an analogy between convective momentum transport and convective heat transport, both of which generally act to accelerate net transport in the direction of diffusion in the uncontrolled setting (e.g. in uncontrolled turbulence). The conjecture holds that convective transport due to flow-field unsteadiness or non-uniformity must continue to accelerate net transport in the direction of diffusion even when control forcing is applied. In the case of heat transport in flows for which viscous heating is negligible, the logical argument for why wall-normal fluid motion must accelerate heat transport in the direction of diffusion is particularly clear. Consider a channel flow with temperature $T$ advecting and diffusing according to the linear equation

$$\frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\nabla T = k\Delta T$$

with boundary conditions $T(x_2 = -1) = -1$ and $T(x_2 = 1) = 1$ and initial conditions $T(t = 0) = x_2$. The fact that $-1 \leqslant T \leqslant 1$ everywhere in the channel for $t > 0$, even when $\phi \neq 0$, follows via the minimum and maximum principles (failure of the minimum principle in this case would lead quickly to a violation of the second law of thermodynamics; the maximum principle follows by symmetry arguments). Near the lower wall, flow-field unsteadiness thus inevitibly transports warmed fluid (with $T > -1$) towards the cold wall (at $T = -1$) and cooled fluid (with $T \approx -1$) away from the cold wall, thereby accelerating the effect of diffusion. (Similarly, near the upper wall, flow-field unsteadiness inevitibly transports cooled fluid (with $T < 1$)

towards the hot wall (at $T = 1$) and hot fluid (with $T \approx 1$) away from the hot wall.) Mathematical proof that the laminar flow indeed provides a fundamental performance limitation in terms of minimizing heat transport is given in Bewley & Ziane (2003).

As shown in the present paper, the same phenomenological justification cannot quite be applied to the problem of momentum transport, as the velocity is governed by a nonlinear equation for which a maximum principle does not apply, and small pockets of 'reverse flow' can occasionally develop in a transient fashion near the walls. The present conjecture simply states that the same result as described above for heat transport must also apply on average for momentum transport.

## 5. Discussion

### 5.1. *Positive aspects of the present results*

In the present work, transient drag reduction to well below the laminar level in a constant mass-flux channel-flow system with zero-net blowing/suction controls has been obtained with a very simple feedback control strategy, (2.1), motivated by global analysis of the nonlinear Navier–Stokes equation (Aamo *et al.* 2003). We have interpreted the physical mechanism responsible for the sublaminar transients as a 'win–win' mechanism in which suction is applied to negative skin-friction regions, and blowing is applied to high positive skin-friction regions, thus intensifying the former and weakening the latter, as illustrated in the flow visualization given in figure 3. The system under consideration has been fully resolved with appropriate grid refinement studies (note the use of grid resolutions of $1024 \times 128$ and box lengths of 60 channel half-widths). The mechanism responsible for the sublaminar drag has been identified as a transient mechanism that probably cannot be exploited to sustain channel-flow drag at sublaminar levels. Note that instantaneous total drag (integrated over the channel walls) in a constant-mass-flow two-dimensional channel flow has in fact been driven to negative levels in the present work; as far as we know, these are the strongest $D(t) < D_L$ transients ever obtained in a blowing/suction-controlled constant mass flux channel-flow system.

### 5.2. *Negative aspects of the present results*

It is sometimes just as important to report negative results as it is to report positive results. Indirect evidence of a negative nature is provided in the present paper. Though we now clearly understand a mechanism which provides strong $D(t) < D_L$ transients (in fact, $D(t) < 0$ transients) in constant mass-flux two-dimensional channel flows, an extensive parametric study of simulations which chain such transients together all indicate the inability of this mechanism to sustain time-averaged drag below laminar levels. The present results thus point consistently towards, but do not prove, the conjecture concerning the possible fundamental performance limitation implied by the laminar flow solution, even in light of recent explorations of flow control strategies demonstrating strong $D(t) < D_L$ transients. As stated previously, mathematical proof of this conjecture remains an open problem.

### 5.3. *Related prior investigations*

The present work grows out of a desire to understand and quantify fundamental performance limitations present in wall-bounded flow systems subject to a broad class of zero-net mass-flux boundary controls (e.g. blowing/suction) or zero-net near-wall forcing on the interior of the flow (e.g. from the Lorentz force arising from wall-mounted magnets and electrodes). We hope that the present numerical evidence will invigorate the investigation of this type of fundamental performance limitiation, perhaps bringing new perspectives to the still unresolved question proposed in § 1 and related fundamental limitations which may be formulated in a similar manner.

To the best of our knowledge, there are currently no definitive analyses of the possible fundamental performance limitation discussed in the present paper, though analyses related to that presented in § 4 for similar problems are presented in, e.g. Doering & Constantin (1992), Constantin & Doering (1994, 1995), Goubet (1996), Fursikov, Gunzburger & Hou (1998) and Keller (2003). On the other hand, there have been a few prior works which imply that the fundamental limitiation proposed in § 1 might be false. We now briefly review these investigations.

In Nosenchuck (1994), a new technique for 'fundamentally restructuring near-wall unsteadiness' with a particular type of Lorentz forcing was introduced, and a proposed mechanism for drag reduction was introduced via the creation of so-called 'two-dimensional fluid rollers' near the wall. Two US patents (nos. 5320309 and 5437421) have been issued to Nosenchuck and Brown in the area of boundary-layer control, both of which are based in part on the concept of 'fundamentally restructuring near-wall unsteadiness' in this fashion using a variety of types of actuation. With this proposed mechanism, when viewed in the appropriate frame of reference (convecting with the vortices), Kelvin–Stuart cat's-eye vortices are claimed to 'insulate the wall from the viscous forces otherwise imparted by the bulk flow', thereby reducing drag and providing a proposed mechanism to sustain the drag at sublaminar levels. In Koumoutsakos (1999), this 'two-dimensional fluid rollers' explanation was provided as a possibly dominant mechanism in the flow resulting from a 'vorticity-flux' control strategy when applied to three-dimensional turbulent channel flow, albeit not providing drag reduction to below the laminar level.

Finally, making concrete the hypothesized existence of some unsteady mechanism to maintain drag at sublaminar levels, such as that described in the previous paragraph, Cortelezzi, Lee, Kim & Speyer (1998), hereinafter referred to as CLKS98, applied a linear control theory to a nonlinear two-dimensional channel flow, and make the claim that a 'dramatic drag reduction was obtained, up to 50% with respect to the laminar flow and 60% with respect to the turbulent flow.' (Note that the use of the phrase 'turbulent flow' in this quote apparently refers to something other than three-dimensional channel-flow turbulence.) It has proved impossible for our group to repeat the CLKS98 result. In the years since, the authors of CLKS98 have not repeated this result either, though we are not aware of any formal retraction of this remarkable CLKS98 claim. Recent results by the same group (see, e.g. Lee *et al.* 2001) have retreated to more modest claims (10–15% drag reduction below the turbulent flow).

The statement of CLKS98 cited above is significant because, if not interpreted correctly, it might lead us to believe that global minimization of drag and global minimization of turbulent kinetic energy (that is, relaminarization) are not equivalent in the control of channel-flow systems, and thus that some unsteady mechanism of 'insulating the wall from the viscous forces otherwise imparted by the bulk flow' is in fact possible. It is for this reason that a critical evaluation of the CLKS98 result is necessary. It is important to determine whether or not we should attempt to drive near-wall flows towards some peculiar unsteady motions rather than towards the laminar state when trying to minimize flow-induced skin-friction drag.

The present numerical results, though not providing a proof of the present conjecture, at least act to illustrate how the CLKS98 result might be consistent with the intuition that the laminar flow solution represents a fundamental limit in the present channel-flow system. In order to be consistent with the present numerical results and with the unproven conjecture stated in § 1, the CLKS98 result might simply reflect an unsustainable transient.

### 5.4. *The larger implications of a study in a two-dimensional channel*

Establishing absolute performance limitations inherent in a certain broad class of flow control problems is an issue of fundamental importance, as it provides us with new insight which is valuable when framing the mathematical statement of the control objective when formulating a model-based feedback control problem. Understanding such a fundamental issue in the two-dimensional setting is an essential prerequisite to understanding the same issue in the three-dimensional setting. In fact, the 'two-dimensional fluid rollers' mechanism for 'insulating the wall from the viscous forces otherwise imparted by the bulk flow', as discussed above, is a completely two-dimensional mechanism which may be critically evaluated in the two-dimensional setting.

The 'win–win' mechanism described in this paper relies on the exploitation of backflow regions near the wall. Note that extensive backflow regions near walls are not naturally occurring in uncontrolled three-dimensional turbulent near-wall flows, and thus we might expect a two-dimensional channel flow to be 'easier' to drive to sublaminar levels than a three-dimensional channel flow with this mechanism. However, this observation is tangential to the subject at hand, which, in fact, is a more fundamental question about a strict bound on the set of solutions that a specific class of Navier–Stokes control problems admits.

It should also be pointed out that it is a trivial matter to initiate largely two-dimensional structures in controlled three-dimensional near-wall flows (even turbulent ones) by synchronizing control forcing on the wall in the spanwise direction. This was, in fact, the (unexpected) result in the vorticity flux control implementation by Koumoutsakos (1999). If a near-wall two-dimensional mechanism for sustained drag reduction to sublaminar levels exists, implementing it in a three-dimensional flow via coordinated control input at the wall should be straightforward.

Further mathematical analysis of the present conjecture, in both the two-dimensional and three-dimensional and steady and unsteady settings, is thus motivated to further our basic phenomenological understanding of the problem of control of wall-bounded flows.

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